

AMBARZUMYAN TYPE THEOREMS ON A TIME SCALE

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ABSTRACT. In this paper, we consider a Sturm–Liouville dynamic equation with Robin boundary conditions on time scale and investigate the conditions which guarantee that the potential function is specified.

1. Introduction

Time scale theory was introduced by Hilger in order to unify continuous and discrete analysis [16]. From then on this approach has received a lot of attention and has applied quickly to various area in mathematics. Sturm–Liouville theory on time scales was studied first by Erbe and Hilger [11] in 1993. Some important results on the properties of eigenvalues and eigenfunctions of a Sturm–Liouville problem on time scales were given in various publications (see e.g. [2], [3], [4], [8]–[10], [12], [14]–[22] and the references therein).

Inverse spectral problems consist in recovering the coefficients of an operator from their spectral characteristics. Although there are vast literature for inverse Sturm–Liouville problems on a continuous interval, there are no study on the general time scales. For Sturm–Liouville operator on a continuous interval, the study which starts inverse spectral theory, was published by Ambarzumyan [1] in 1929. He prove that: if q is continuous function on $(0, 1)$ and the eigenvalues of the problem

$$\begin{aligned} -y'' + q(t)y &= \lambda y, \quad t \in (0, 1) \\ y'(0) &= y'(1) = 0 \end{aligned}$$

are given as $\lambda_n = n^2\pi^2$, $n \geq 0$ then $q \equiv 0$.

Freiling and Yurko [13] generalized this result as $\lambda_0 = \int_0^1 q(t)dt$ implies $q \equiv \lambda_0$.

The goal of this paper to prove an Ambarzumyan type theorem on a general time scale and to apply it on the a special time scale. In our main result, Theorem 1, we generalize the results of Freiling and Yurko for Sturm–Liouville operator with more general boundary conditions on a time scale.

2. Preliminaries and Main Results

If \mathbb{T} is a closed subset of \mathbb{R} it called as a time scale. The jump operators σ , ρ and graininess operator on \mathbb{T} are defined as follow:

$$\begin{aligned} \sigma : \mathbb{T} &\rightarrow \mathbb{T}, \quad \sigma(t) = \inf \{s \in \mathbb{T} : s > t\} \text{ if } t \neq \sup \mathbb{T}, \\ \rho : \mathbb{T} &\rightarrow \mathbb{T}, \quad \rho(t) = \sup \{s \in \mathbb{T} : s < t\} \text{ if } t \neq \inf \mathbb{T}, \\ \sigma(\sup \mathbb{T}) &= \sup \mathbb{T}, \quad \rho(\inf \mathbb{T}) = \inf \mathbb{T}, \\ \mu : \mathbb{T} &\rightarrow [0, \infty) \quad \mu(t) = \sigma(t) - t. \end{aligned}$$

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A point of \mathbb{T} is called as left-dense, left-scattered, right-dense, right-scattered and isolated if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$ and $\rho(t) < t < \sigma(t)$, respectively.

A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called rd-continuous on \mathbb{T} if it is continuous at all right-dense points and has left-sided limits at all left-dense points in \mathbb{T} . The set of rd-continuous functions on \mathbb{T} is denoted by $C_{rd}(\mathbb{T})$ or C_{rd} .

$$\text{Put } \mathbb{T}^k := \begin{cases} \mathbb{T} - \{\sup \mathbb{T}\}, & \text{sup } \mathbb{T} \text{ is left-scattered} \\ \mathbb{T}, & \text{the other cases} \end{cases}, \quad \mathbb{T}^{k^2} := (\mathbb{T}^k)^k.$$

Let $t \in \mathbb{T}^k$. Suppose that for given any $\varepsilon > 0$, there exist a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in U$ then, f is called differentiable at $t \in \mathbb{T}^k$. We call $f^\Delta(t)$ the delta derivative of f at t . A function $F : \mathbb{T} \rightarrow \mathbb{R}$ defined as $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^k$ is called an antiderivative of f on \mathbb{T} . In this case the Cauchy integral of f is defined by

$$\int_a^b f(t) \Delta t = F(b) - F(a), \text{ for } a, b \in \mathbb{T}.$$

Some important relations whose proofs appear in [6], chapter1 will be needed. We collect them in the following lemma.

Lemma 1. *Let $f : \mathbb{T} \rightarrow \mathbb{R}$, $g : \mathbb{T} \rightarrow \mathbb{R}$ be two functions and $t \in \mathbb{T}^k$.*

- i) If $f^\Delta(t)$ exists, then f is continuous at t ;*
- ii) if t is right-scattered and f is continuous at t , then f is differentiable at t and $f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\sigma(t) - t}$, where $f^\sigma(t) = f(\sigma(t))$;*
- iii) if $f^\Delta(t)$ exists, then $f^\sigma(t) = f(t) + \mu(t)f^\Delta(t)$;*
- iv) if $f^\Delta(t), g^\Delta(t)$ exist and $(fg)(t)$ is defined, then $(fg)^\Delta(t) = (f^\Delta g + f^\sigma g^\Delta)(t)$ and if $(gg^\sigma)(t) \neq 0$, then $\left(\frac{f}{g}\right)^\Delta(t) = \left(\frac{f^\Delta g - f g^\Delta}{gg^\sigma}\right)(t)$;*
- v) if $f \in C_{rd}(\mathbb{T})$, then it has an antiderivative on \mathbb{T} ;*
- vi) if \mathbb{T} consists of only isolated points and $a, b \in \mathbb{T}$ with $a < b$, then $\int_a^b f(t) \Delta t = \sum_{t \in [a, b) \cap \mathbb{T}} \mu(t)f(t)$;*
- vii) if $f(t) \geq 0$ for all $t \in [a, b] \cap \mathbb{T}$ and $\int_a^b f(t) \Delta t = 0$, then $f(t) \equiv 0$.*

Throughout this paper we assume that \mathbb{T} is a bounded time scale, $a = \inf \mathbb{T}$ and $b = \sup \mathbb{T}$. Consider the boundary value problem $L = L(q, h_a, h_b)$ generated by the Sturm-Liouville dynamic equation

$$(1) \quad \ell y := -y^{\Delta\Delta}(t) + q(t)y^\sigma(t) = \lambda y^\sigma(t), \quad t \in \mathbb{T}^{k^2}$$

subject to the boundary conditions

$$(2) \quad y^\Delta(a) - h_a y(a) = 0$$

$$(3) \quad y^\Delta(\rho(b)) - h_b y(\rho(b)) = 0$$

where $q(t)$ is real valued continuous function on \mathbb{T} , $h_a, h_b \in \mathbb{R}$ and λ is the spectral parameter. Additionally, we assume that $a \neq \rho(b)$, $1 + h_a \mu(a) \neq 0$ and $1 + h_b \mu(\rho(b)) \neq 0$.

Definition 1. *The values of the parameter for which the equation (1) has nonzero solutions satisfy (2) and (3), are called eigenvalues and the corresponding nontrivial solutions are called eigenfunctions.*

It is proven in [6] that all eigenvalues of the problem (1)-(3) are real numbers.

Definition 2. A solution y of (1) is said to have a zero at $t \in \mathbb{T}$ if $y(t) = 0$, and it has a node between t and $\sigma(t)$ if $y(t)y(\sigma(t)) < 0$. A generalized zero of y is then defined as a zero or a node.

Lemma 2 ([2]). The eigenvalues of (1)-(3) may be arranged as $-\infty < \lambda_1 < \lambda_2 < \lambda_3 < \dots$ and an eigenfunction corresponding to λ_{k+1} has exactly k generalized zeros in the open interval (a, b) .

Lemma 3. If $y(t)$ is an eigenfunction of the problem (1)-(3) then $y^\sigma(a) \neq 0$ and $y^\sigma(\rho(b)) \neq 0$.

Proof. It is clear from Lemma 1 that $y^\sigma(a) = y(a) + \mu(a)y^\Delta(a) = y(a)[1 + h_a\mu(a)]$ and $y^\sigma(\rho(b)) = y(\rho(b)) + \mu(\rho(b))y^\Delta(\rho(b)) = y(\rho(b))[1 + h_b\mu(\rho(b))]$. We claim that $y(a) \neq 0$ and $y(\rho(b)) \neq 0$. Otherwise, from (2) and (3) $y^\Delta(a) = 0$ or $y^\Delta(\rho(b)) = 0$ hold, then by the uniqueness theorem of the solution of initial value problems $y(t)$ is identically vanish which contradicts that it is the eigenfunction. Therefore the proof is completed from the assumption $1 + h_a\mu(a) \neq 0$, $1 + h_b\mu(\rho(b)) \neq 0$. \square

Theorem 1. Let λ_1 be the first eigenvalue of (1)-(3). If

$$\lambda_1 \geq \frac{1}{\rho(b) - a} \left\{ h_a - h_b + \int_a^{\rho(b)} q(t) \Delta t \right\},$$

then $q(t) \equiv \lambda_1$.

Proof. Let $y_1(t)$ be the corresponding eigenfunction to λ_1 . From eq(1) and Lemma 2 we can write on \mathbb{T}^{k^2}

$$\frac{y_1^{\Delta\Delta}(t)}{y_1^\sigma(t)} = q(t) - \lambda_1.$$

It is from the relation

$$\frac{y_1^{\Delta\Delta}(t)}{y_1^\sigma(t)} = \frac{[y_1^\Delta(t)]^2}{y_1^\sigma(t)y_1(t)} + \left[\frac{y_1^\Delta(t)}{y_1(t)} \right]^\Delta$$

that

$$\left[\frac{y_1^\Delta(t)}{y_1(t)} \right]^\Delta = q(t) - \lambda_1 - \frac{[y_1^\Delta(t)]^2}{y_1^\sigma(t)y_1(t)}.$$

From Lemma 3 we can integration of both sides from a to $\rho(b)$. Therefore the following equality is obtained

$$\begin{aligned} \int_a^{\rho(b)} \frac{[y_1^\Delta(t)]^2}{y_1^\sigma(t)y_1(t)} \Delta t &= \frac{y_1^\Delta(a)}{y_1(a)} - \frac{y_1^\Delta(\rho(b))}{y_1(\rho(b))} + \int_a^{\rho(b)} [q(t) - \lambda_1] \Delta t \\ &= h_a - h_b + \int_a^{\rho(b)} q(t) \Delta t - \lambda_1 (\rho(b) - a). \end{aligned}$$

It can be seen from Lemma 2 and our hypothesis that the right side of the last equality is negative and the left side is non-negative. Thus $y_1^\Delta(t) \equiv 0$ and so $y_1(t)$ is constant. Substituting $y_1(t)$ is constant into equation (1), it is concluded that $q(t) \equiv \lambda_1$. \square

Corollary 1. *The first eigenvalue of the problem $-y^{\Delta\Delta} + q(t)y^\sigma = \lambda y^\sigma$, $y^\Delta(a) = y^\Delta(\rho(b)) = 0$ is $\lambda_1 = \frac{1}{\rho(b)-a} \int_a^{\rho(b)} q(t)\Delta t$ then, $q(t) \equiv \lambda_1$.*

This corollary is a generalization of the results of Freiling and Yurko [13] onto the time scale.

Corollary 2. *Under the hypothesis $\int_a^{\rho(b)} q(t)\Delta t = 0$; if $q(t) \neq 0$, then the problem*

$$\begin{aligned} -y^{\Delta\Delta} + q(t)y^\sigma &= \lambda y^\sigma, \\ y^\Delta(a) = y^\Delta(\rho(b)) &= 0 \end{aligned}$$

has at least one negative eigenvalue.

We conclude this paper with specializing our first result for a particular time scale which consists of only isolated points.

Remark 1. *Consider the time scale*

$$\mathbb{T} = \{x_k \in \mathbb{R} : a = x_0 < x_1 < x_2 < \dots < x_n = b\}$$

and the following problem

$$\begin{aligned} -y^{\Delta\Delta} + q(t)y^\sigma &= \lambda y^\sigma, \\ y^\Delta(a) = y^\Delta(\rho(b)) &= 0. \end{aligned}$$

If the first eigenvalue of the problem satisfy $\lambda_1 \geq \max\{q(t) : t \in \mathbb{T}\}$, then $q(t) \equiv \lambda_1$.

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